Marmara University
Faculty of Engineering
Math 101.1/Math 1001.1
Midterm Exam

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Date: 30/03/2016

Question:	1	2	3	4	5	6	Total
Points:	10	15	15	10	30	20	100
Score:							

NAME: ______________________

SIGNATURE: ______________________

STUDENT NO: ______________________

Give detailed work. Write your final answers in the box provided.

1. (10 points) Find the equation of the tangent line at the point $(-1,2)$ to the graph of the curve

$x^3 - 5x^2y^2 + 2y^2 + 5 = 0$ → should be -8, But does not change the solution.

Taking $\frac{d}{dx}$ of both sides

$3x^2 - 10xy^2 - 10x^2y\,y' + 4yy' = 0$

At $x=-1$, $y=2$,

$3 + 40 - 20y' + 8y' = 0$

slope $= y' = 43/12$

$$\boxed{y - 2 = \frac{43}{12}(x+1)}$$

2. Differentiate the following functions. Do not simplify your answer.

(a) (5 points) $y = \dfrac{\arctan(x)}{x}$

$$\boxed{y' = \frac{\frac{1}{x^2+1}\cdot x - \arctan(x)}{x^2}}$$

(b) (5 points) $y = \ln\left(x + e^{-3x}\right)$

$$\boxed{y' = \frac{1}{x+e^{-3x}}\cdot\left(1 - 3e^{-3x}\right)}$$

(c) (5 points) $y = 3^x x^3$

$$\boxed{y' = 3^x \ln 3\, x^3 + 3^x\, 3x^2}$$

3. **(15 points) Find the derivative of**

$$y = \left(\frac{1}{x}\right)^{\ln x}$$

at $x = e$.

Take ln of both sides:

$$\ln y = \ln\left(\frac{1}{x}\right)^{\ln x} = \ln x \ \ln\left(\frac{1}{x}\right) = \ln x\left[-\ln x\right] = -\left[\ln x\right]^2$$

$$\frac{y'}{y} = -2\ln x \cdot \frac{1}{x}$$

$$y' = -2\left(\frac{1}{x}\right)^{\ln x} \ln x \cdot \frac{1}{x}$$

$$x = e \Rightarrow y' = -2\left(\frac{1}{e}\right)^{1} \cdot 1 \cdot \frac{1}{e}$$

$$\boxed{-2/e^2}$$

4. **(10 points) Use the Mean Value Thorem to show that for any two real numbers a and b,**

$$|\cos a - \cos b| \leq |a - b|$$

Let $f(x) = \cos x$

If $a = b$, the problem is trivial.

Suppose $a < b$.

By Mean Value Theorem, there is a number c

$a < c < b$ and

$$f(b) - f(a) = f'(c)(b-a)$$

$$\cos b - \cos a = (-\sin c)(b-a)$$

$$|\cos b - \cos a| = |-\sin c|\,|b-a| \leq |b-a|.$$

5. Find the following limits. (Do not use L'Hopital's Rule)

(a) (5 points)

$$\lim_{x\to 6+} \frac{(x-5)(3-x)}{(x-6)(x-1)}$$

if $x>6$ then $\frac{(x-5)(3-x)}{(x-6)(x-1)} < 0$

As $x \to 6$, $(x-5)(3-x) \to -3$, $(x-6)(x-1) \to 0$

$\boxed{-\infty}$

(b) (10 points)

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$$

$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$

$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$

$\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$

By Sandwich Theorem, $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$

$\boxed{0}$

(c) (10 points)

$$\lim_{x\to\infty} \sqrt{x^2+x} - x = \lim_{x\to\infty} \frac{(\sqrt{x^2+x}-x)(\sqrt{x^2+x}+x)}{\sqrt{x^2+x}+x}$$

$$= \lim_{x\to\infty} \frac{x^2+x-x^2}{|x|\sqrt{1+1/x}+x} = \lim_{x\to\infty} \frac{\cancel{x}}{\cancel{x}\sqrt{1+1/x}+\cancel{x}}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{1+\underbrace{1/x}_{\to 0}}+1} = \frac{1}{1+1}$$

$\boxed{2}$

(d) (5 points)

$$\lim_{h\to 0} \frac{(2+h)\ln(2+h) - 2\ln 2}{h}$$

(**Hint**: relate this limit to the limit definition of derivative of a function.)

$f(x) = x \ln x$

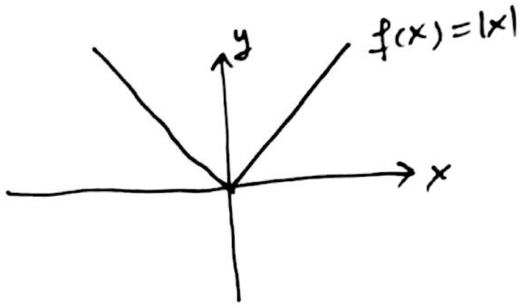
$f'(2) = \lim_{h\to 0} \frac{(2+h)\ln(2+h) - 2\ln 2}{h}$

$f'(2) = \left[1 \ln x + x \cdot \frac{1}{x}\right]\Big|_{x=2} = \ln 2 + 1$

$\boxed{\ln 2 + 1}$

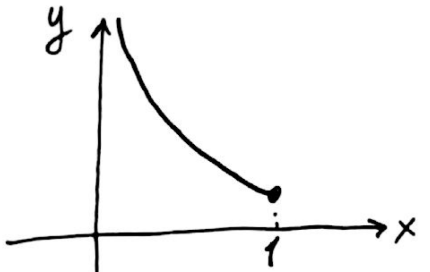
6. Give the formula of a function which satisfies the given conditions. If such a function does not exist, explain the reason.

   (a) (5 points) f is continuous at $x = 0$, but f is not differentiable at $x = 0$.



$f(x) = |x|$

   (b) (5 points) f is continuous on $(0, 1]$ but does not achieve its maximum on $(0, 1]$.



$f(x) = \frac{1}{x}$

   (c) (5 points) f is continuous on $[0, 1]$ but does not achieve its maximum on $[0, 1]$.

See Lecture Notes, pg. 27.

Such a function does not exist by Extreme Value theorem.

   (d) (5 points) f has a limit at $x = 1$ but it is not continuous at $x = 1$.

$$f(x) = \begin{cases} x, & \text{if } x \neq 1 \\ 2, & \text{if } x = 1. \end{cases}$$